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Additive idempotence preservers ☆

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Abstract

Let $\mathscr{A}$ be a local matrix algebra over a field of characteristic different from 2, $\mathscr{B}$ an arbitrary algebra, and $\mathscr{X}, \mathscr{Y}$ Banach spaces. Additive mappings $\Phi : \mathscr{A} \to \mathscr{B}$, which preserve idempotents are characterized. The result is then used in classifying additive surjections $\Phi : \mathscr{B}(\mathscr{X}) \to \mathscr{B}(\mathscr{Y})$, which preserve idempotents and do not annihilate finite-rank operators.

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1. Introduction

One of the open problems in linear algebra is to characterize all unital, linear bijections $\Phi : \mathscr{A} \to \mathscr{B}$ between semisimple Banach algebras $\mathscr{A}$ and $\mathscr{B}$ that preserve invertibility. It is conjectured that any such mapping must be a Jordan isomorphism, i.e., that it must satisfy $\Phi(a^2) = \Phi(a)^2$ (see [11]). In the past decade, different techniques have been introduced by different authors in an attempt to solve the problem. For example, in [4] the authors deduced that Φ preserves idempotents and were then able to validate the conjecture for the case of Φ, which maps a von Neumann algebra $\mathscr{A}$ onto $\mathscr{B}(\mathscr{Y})$, an algebra of bounded operators on a complex Banach space. The same technique was later extended in [1]; see also [6].

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Mappings, which preserve idempotents were fruitfully used in yet another problem of characterizing local automorphisms, i.e., linear, surjective mappings $\Phi : \mathscr{B}(\mathscr{X}) \to \mathscr{B}(\mathscr{X})$ such that for any $a \in \mathscr{B}(\mathscr{X})$ there exists an automorphism Φ_a on $\mathscr{B}(\mathscr{X})$ with $\Phi(a) = \Phi_a(a)$ (see [3]).

While linear preserves are comparatively well understood, much less is known about additive ones, especially in the context of infinite dimensional algebras. It is our aim to explore this fertile topic in an additive context, thus proving that any surjective idempotence-preserver between $\mathscr{B}(\mathscr{X})$ and $\mathscr{B}(\mathscr{Y})$ (with $\mathscr{X}, \mathscr{Y}$ infinite-dimensional Banach spaces) is 'almost' Jordan if it does not annihilate rank-one idempotents (see the Main Theorem and Remark 3.4). This result was obtained by studying additive idempotence-preserves, which map a local matrix algebra $\mathscr{A}$ *into* an arbitrary algebra $\mathscr{B}$. We acknowledge that some ideas are borrowed from [5,12].

2. Mappings defined on $\mathscr{M}_n(\Delta)$

In the present section, additive mappings $\Phi : \mathscr{M}_n(\Delta) \to \mathscr{B} := \mathrm{Hom}_{\Delta'}(\mathscr{X})$ that preserve idempotents (i.e., $p^2 = p \Rightarrow \Phi(p)^2 = \Phi(p)$) will be studied. Here $\mathscr{M}_n(\Delta)$ is the algebra of $n \times n$ matrices with entries from a field Δ and $\mathscr{X}$ a left module over a field Δ'. It will be assumed that $\mathrm{char}(\Delta) \neq 2$. We emphasize that $\mathscr{M}_n(\Delta)$ is spanned by the elementary matrices e_{ij}, which have 1 in the (i, j) entry and zeros elsewhere. Repeatedly, its elements will be regarded as matrix representations of module homomorphisms on a left Δ-module $\mathscr{N}$ and relative to a fixed basis $\mathbf{n}_1, \dots, \mathbf{n}_n$. Having $\mathbf{x} \in \mathscr{N}$ and a Δ-linear functional f on $\mathscr{N}$, a *rank-one operator* that maps $\mathbf{n} \in \mathscr{N}$ to $f(\mathbf{n})\mathbf{x}$ will be denoted by $\mathbf{x} \otimes f$. As usual, $\mathscr{U} \otimes_R \mathscr{V}$ will stand for the tensor product of R-modules and a^{t} for the transposition of the matrix a. Finally, $\mathbb{F} := \mathbb{F}(\Delta)$ will denote the *prime field* of Δ (i.e., the field generated by 1), and Id *the identity mapping*, while $\mathscr{P}(\mathscr{A}) := \mathrm{Lin}_{\mathbb{F}}\{p \in \mathscr{A};\ p^2 = p\}$; $\mathscr{A}$ being a Δ-algebra. Throughout this section, the abbreviation $\mathscr{P} := \mathscr{P}(\mathscr{M}_n(\Delta))$ will be used.

Since Φ is additive, a moment's thought implies that $\mathrm{char}(\Delta) = \mathrm{char}(\Delta')$, or else it vanishes identically. In the latter triviality Φ is Jordan, and in the rest of this section we will prove the same for the former possibility, where $\Phi \neq 0$ and $\mathbb{F}(\Delta) = \mathbb{F}(\Delta') =:$ $\mathbb{F}$. Note that such Φ is necessarily linear over $\mathbb{F}$. More information about Jordan mappings can be found in [7]; for instance, any Jordan Ψ satisfies $\Psi(ab + ba) = \Psi(a)\Psi(b) + \Psi(b)\Psi(a)$, and also

$$\Psi(aba) = \Psi(a)\Psi(b)\Psi(a), \tag{1}$$

$$\Psi(abc + cba) = \Psi(a)\Psi(b)\Psi(c) + \Psi(c)\Psi(b)\Psi(a). \tag{2}$$

(Proof: $aba \pm bab = (b \pm a)^3 - b^3 \mp a^3 \mp (b^2a + ab^2) - (a^2b + ba^2)$ gives (1) via $x^3 = 1/2(x^2x + xx^2)$, while $abc + cba = (a + c)b(a + c) - aba - cbc$ gives the last equation.)

Having fixed the notation we proceed with the first lemma. Let us just remark that each Δ'-module $\mathscr{X}$ has a Hamel basis the elements of which can be well-ordered, so each module homomorphism can be represented—relative to the fixed Hamel basis—by a (possibly infinite) matrix.

Lemma 2.1 (Diagonalization lemma). *There exist Δ'–submodules $\hat{\mathscr{Y}}, \hat{\mathscr{Z}}; \mathscr{U}$ of $\mathscr{X}$ and a decomposition $\mathscr{X} \simeq (\hat{\mathscr{Y}} \otimes_{\mathbb{F}} \mathrm{Lin}_{\mathbb{F}}\{\mathbf{n}_1, \ldots, \mathbf{n}_n\}) \oplus (\hat{\mathscr{Z}} \otimes_{\mathbb{F}} \mathrm{Lin}_{\mathbb{F}}\{\mathbf{n}_1, \ldots, \mathbf{n}_n\}) \oplus \mathscr{U}$ such that $\Phi(a) = (\mathrm{Id}_{\hat{\mathscr{Y}}} \otimes_{\mathbb{F}} a) \oplus (\mathrm{Id}_{\hat{\mathscr{Z}}} \otimes_{\mathbb{F}} a^{\mathrm{t}}) \otimes 0$ whenever $a \in \mathscr{M}_n(\mathbb{F})$.*

Proof. Regard $\mathscr{B}$ and $\mathscr{M}_n(\mathbb{F})$ as $\mathbb{F}$-algebras. By [3, Theorem 2.1] *the restriction* $\Phi|_{\mathscr{M}_n(\mathbb{F})} = \phi \dot{\oplus} \tau$; i.e., it is an orthogonal sum of the $\mathbb{F}$-linear homomorphism ϕ and of the $\mathbb{F}$-linear antihomomorphisms τ. Let $P := \phi(\mathrm{Id})$ and $Q := \tau(\mathrm{Id})$ be two orthogonal idempotents; define $\mathscr{U} := \mathrm{Ker}(P + Q)$. Clearly then,

$$\mathscr{X} = \mathrm{Im}P \oplus \mathrm{Im}Q \oplus \mathscr{U}. \tag{3}$$

Furthermore, since P and Q are identities in $\phi(\mathscr{M}_n(\mathbb{F}))$ and in $\tau(\mathscr{M}_n(\mathbb{F}))$, respectively, we easily derive that $\Phi(a)|_{\mathscr{U}} \equiv 0$, and $\phi(a)\mathrm{Im}P \subseteq \mathrm{Im}P$, and $\tau(a)\mathrm{Im}Q \subseteq \mathrm{Im}Q$, and that

$$\Phi(a) = (\phi(a)|_{\mathrm{Im}P}) \oplus (\tau(a)|_{\mathrm{Im}Q}) \oplus 0 \tag{4}$$

is a block-diagonal for $a \in \mathscr{M}_n(\mathbb{F})$.

Define $\hat{\mathscr{Y}} := \mathrm{Im}\phi(e_{11}) \leqslant \mathrm{Im}P$ and choose a Δ-linear functional f_1 on $\mathscr{N}$ such that the matrix representation of $\mathbf{n}_1 \otimes f_1$ equals e_{11} on the predefined basis $\{\mathbf{n}_1, \ldots, \mathbf{n}_n\}$ of $\mathscr{N}$. Clearly, the additive (hence $\mathbb{F}$-linear!) mapping

$$T : \hat{\mathscr{Y}} \otimes_{\mathbb{F}} \mathrm{Lin}_{\mathbb{F}}\{\mathbf{n}_1, \ldots, \mathbf{n}_n\} \to \mathrm{Im}P, \quad T(\hat{\mathbf{y}} \otimes_{\mathbb{F}} \mathbf{m}) := \phi(\mathbf{m} \otimes f_1)\hat{\mathbf{y}}$$

is well-defined. It is also injective: namely, if $T(\sum_i \hat{\mathbf{y}}_i \otimes_{\mathbb{F}} \mathbf{m}_i) = \sum_i \phi(\mathbf{m}_i \otimes f_1)\hat{\mathbf{y}}_i = 0$ for certain $\hat{\mathbf{y}}_i \in \hat{\mathscr{Y}}$ and $\mathbf{m}_i := \alpha_{i1}\mathbf{n}_1 + \cdots + \alpha_{in}\mathbf{n}_n \in \mathrm{Lin}_{\mathbb{F}}\{\mathbf{n}_1, \ldots, \mathbf{n}_n\}$ (where $\alpha_{ij} \in \mathbb{F}$), then

$$\begin{aligned} 0 &= \sum_i \phi(\mathbf{m}_i \otimes f_1)\hat{\mathbf{y}}_i = \sum_i \sum_j \alpha_{ij}\phi(\mathbf{n}_j \otimes f_1)\hat{\mathbf{y}}_i = \sum_{i,j} \alpha_{ij}\phi(e_{j1})\hat{\mathbf{y}}_i \\ &= \sum_{i,j} \phi(e_{j1})(\alpha_{ij}\hat{\mathbf{y}}_i). \end{aligned} \tag{5}$$

Multiplying (5) by $\phi(e_{11})$ gives $0 = \sum_{i,j} \phi(e_{11})\phi(e_{j1})(\alpha_{ij}\hat{\mathbf{y}}_i) = \sum_{i,j} \phi(e_{11}e_{j1})(\alpha_{ij}\hat{\mathbf{y}}_i) = \sum_i \phi(e_{11})(\alpha_{i1}\hat{\mathbf{y}}_i)$. Since $\phi(e_{11})$ is an identity on $\hat{\mathscr{Y}}$, we derive $\sum_i \alpha_{i1}\hat{\mathbf{y}}_i = 0$. Multiplying (5) by $\phi(e_{12}), \ldots, \phi(e_{1n})$ in succession gives similarly $\sum_i \alpha_{i2}\hat{\mathbf{y}}_i = 0 = \cdots = \sum_i \alpha_{in}\hat{\mathbf{y}}_i$. Therefore, $\sum_i \hat{\mathbf{y}}_i \otimes_{\mathbb{F}} \mathbf{m}_i = \sum_j (\sum_i \alpha_{ij}\hat{\mathbf{y}}_i) \otimes_{\mathbb{F}} \mathbf{n}_j = 0$, as anticipated.

Moreover, T is onto. Namely, given $\mathbf{y} \in \mathrm{Im}P$ we have

$$\begin{aligned}\mathbf{y} = P\mathbf{y} = \phi(\mathrm{Id})\mathbf{y} &= \sum_i \phi(e_{ii})\mathbf{y} = \sum_i \phi(e_{i1}e_{11}e_{1i})\mathbf{y}\\ &= \sum_i \phi(e_{i1})\phi(e_{11})\phi(e_{1i})\mathbf{y}\\ &= \sum_i \phi(\mathbf{n}_i \otimes f_1)(\phi(e_{11})\phi(e_{1i})\mathbf{y})\\ &= \sum_i T((\phi(e_{11})\phi(e_{1i})\mathbf{y}) \otimes_{\mathbb{F}} \mathbf{n}_i) \in \mathrm{Im}T.\end{aligned}$$

Lastly, pick arbitrary $\hat{\mathbf{y}} \in \hat{\mathscr{Y}}$, $a \in \mathscr{M}_n(\mathbb{F})$ and $\mathbf{m} \in \mathrm{Lin}_{\mathbb{F}}\{\mathbf{n}_1, \ldots, \mathbf{n}_n\}$. Then

$$\begin{aligned}\phi(a)T(\hat{\mathbf{y}} \otimes_{\mathbb{F}} \mathbf{m}) &= \phi(a)\phi(\mathbf{m} \otimes f_1)\hat{\mathbf{y}} = \phi(a \cdot (\mathbf{m} \otimes f_1))\hat{\mathbf{y}}\\ &= \phi((a\mathbf{m}) \otimes f_1)\hat{\mathbf{y}} = T(\hat{\mathbf{y}} \otimes_{\mathbb{F}} a\mathbf{m})\\ &= T(\mathrm{Id}_{\mathscr{Y}} \otimes_{\mathbb{F}} a)(\hat{\mathbf{y}} \otimes_{\mathbb{F}} \mathbf{m}),\end{aligned}$$

which gives $\phi(a)T = T(\mathrm{Id}_{\mathscr{Y}} \otimes_{\mathbb{F}} a)$, so that $\phi(a)|_{\mathrm{Im}P} = \phi(a)|_{\mathrm{Im}T} = T(\mathrm{Id}_{\mathscr{Y}} \otimes_{\mathbb{F}} a)T^{-1}$. In a similar way (or else by considering *homomorphism* $a \mapsto \tau(a^{\mathrm{t}})$) one shows that $\mathrm{Im}Q \simeq \hat{\mathscr{Z}} \otimes_{\mathbb{F}} \mathrm{Lin}_{\mathbb{F}}\{\mathbf{n}_1, \ldots, \mathbf{n}_n\}$, and that one can identify $\tau(a)|_{\mathrm{Im}Q}$ by $\mathrm{Id}_{\hat{\mathscr{Z}}} \otimes_{\mathbb{F}} a^{\mathrm{t}}$, where $\hat{\mathscr{Z}} := \mathrm{Im}\tau(e_{11})$. Eqs. (3) and (4) finish the proof. □

Remark 2.2. In the sequel, no distinction will be made between $\hat{\mathscr{Y}} \otimes_{\mathbb{F}} \mathrm{Lin}_{\mathbb{F}}\{\mathbf{n}_1, \ldots, \mathbf{n}_n\}$ and $\hat{\mathscr{Z}} \otimes_{\mathbb{F}} \mathrm{Lin}_{\mathbb{F}}\{\mathbf{n}_1, \ldots, \mathbf{n}_n\}$ on one hand, and submodules $\mathrm{Im}P =: \mathscr{Y}$, $\mathrm{Im}Q =: \mathscr{Z}$, on the other.

By Eq. (3), $\mathscr{X} = \mathscr{Y} \oplus \mathscr{Z} \oplus \mathscr{U}$. Moreover, by choosing a well-ordered Hamel basis $(\hat{y}_\imath)_\imath$ for $\hat{\mathscr{Y}}$ and $(\hat{z}_\jmath)_\jmath$ for $\hat{\mathscr{Z}}$, respectively, Eq. (4) implies $\Phi(a) = (\oplus_\imath a) \oplus (\oplus_\jmath a^{\mathrm{t}}) \oplus 0$, $(a \in \mathscr{M}_n(\mathbb{F}))$—a (possibly infinite) block-diagonal matrix, where each block is of the $n \times n$ dimension.

Lemma 2.3. *In addition to the previous lemma, if $a \in \mathscr{P}$ then $\Phi(a)|_{\mathscr{U}} = 0$ and $\mathrm{Im}\Phi(a) \leqslant \mathscr{Y} \oplus \mathscr{Z}$. Consequently, $\Phi(a) = \Phi'(a) \oplus 0$ and $\Phi'(a)$ can be represented by a (possibly infinite) block matrix, where each block is of the $n \times n$ dimension. Furthermore, in each column there can only be a finite number of nonzero blocks.*

Proof. It is enough to check this in case a is an idempotent. Then, however, the sum of idempotents a and $(\mathrm{Id} - a)$ is again an idempotent. A straightforward argument reveals that $\Phi(a)\Phi(\mathrm{Id} - a) = 0 = \Phi(\mathrm{Id} - a)\Phi(a)$ (see also [1, p. 922]). This readily implies that $\Phi(a)\Phi(\mathrm{Id}) = \Phi(a) = \Phi(\mathrm{Id})\Phi(a)$, so $\mathrm{Im}\Phi(a) \leqslant \mathrm{Im}\Phi(\mathrm{Id}) = \mathscr{Y} \oplus \mathscr{Z}$ and $\mathscr{U} \leqslant \mathrm{Ker}\Phi(a)$. The rest follows easily considering Remark 2.2 and the definition of Hamel basis. □

Corollary 2.4. *If an additive $\Phi : \mathscr{M}_n(\Delta) \to \mathscr{M}_k(\Delta')$ preserves idempotents and $n > k$, then $\Phi|_{\mathscr{P}} = 0$.*

Proof. The assumption imply $\mathscr{Y} = 0 = \mathscr{Z}$, so $\mathscr{X} = \mathscr{U}$. $\square$

The following observation will play a crucial role in the sequel.

Lemma 2.5. *Φ maps each rank-one nilpotent into a nilpotent element of nilindex 2 or less. Consequently, if a, b, $(a+b)$ are all rank-one nilpotents, then $\Phi(a)\Phi(b) + \Phi(b)\Phi(a) = 0$.*

Proof. Suppose $n := \mathbf{x} \otimes f$ is a rank-one nilpotent. There exists $\mathbf{x}_1 \in \mathscr{N}$ with $f(\mathbf{x}_1) = 1$. Therefore, both $p := \mathbf{x}_1 \otimes f$, as well as $p \pm n$ are idempotents which imply that

$$\begin{aligned}\Phi(p) \pm \Phi(n) &= \Phi(p \pm n)^2\\ &= \Phi(p) \pm (\Phi(p)\Phi(n) + \Phi(n)\Phi(p)) + \Phi(n)^2.\end{aligned} \tag{6}$$

By summing the equations in (6) we get $2\Phi(n)^2 = 0$; hence $\Phi(n)^2 = 0$ since there are no elements of order 2 in $\mathscr{B}$ (i.e., $x + x = 0 \Rightarrow x = 0$). The rest is straightforward. $\square$

Remark 2.6. It follows from the proof that each rank-one nilpotent can be written as a difference of two idempotents of the rank one. Thus, each rank-one nilpotent is in $\mathscr{P}$.

Lemma 2.7. *Suppose $n \geqslant 3$, and assume the decomposition of $\mathscr{X}$ from Remark 2.2. If $\delta \in \Delta$ and if $m \in \mathscr{M}_n(\mathbb{F})$ is a rank-one nilpotent, then*

$$\Phi(\delta m) = \begin{pmatrix} A & \\ & B \end{pmatrix} \oplus 0,$$

where $A = (A_{\hat{\imath}\hat{\imath}})_{\hat{\imath}}$ and $B = (B_{\hat{\jmath}\hat{\jmath}})_{\hat{\jmath}}$ are (possibly infinite) block matrices. Moreover, blocks constituting A and B are of the form $A_{\hat{\imath}\hat{\imath}} = \mathfrak{a}_{\hat{\imath}\hat{\imath}} m$ and $B_{\hat{\jmath}\hat{\jmath}} = \mathfrak{b}_{\hat{\jmath}\hat{\jmath}} m^{\mathrm{t}}$, with $\mathfrak{a}_{\hat{\imath}\hat{\imath}}, \mathfrak{b}_{\hat{\jmath}\hat{\jmath}} \in \Delta'$, respectively.

Proof. First, we check the validity of the claims in a special case when $m = e_{1n}$. By Lemma 2.3,

$$\Phi(\delta e_{1n}) = \begin{pmatrix} A & U \\ V & B \end{pmatrix}$$

(we omitted the trailing zero!), so it remains to see that $U = 0 = V$ and that the constitutents of A (respectively, B) are of the right form. To verify this we write $A_{\hat{\imath}\hat{\imath}} := ({}_{\hat{\imath}\hat{\imath}}\mathfrak{a}_{kl})_{1 \leqslant k,l \leqslant n} \in \mathscr{M}_n(\Delta')$, $B_{\hat{\jmath}\hat{\jmath}} := ({}_{\hat{\jmath}\hat{\jmath}}\mathfrak{b}_{kl})_{1 \leqslant k,l \leqslant n}$; and similarly for $U = (U_{\hat{\imath}\hat{\jmath}})_{\hat{\imath}\hat{\jmath}}$ with $U_{\hat{\imath}\hat{\jmath}} := ({}_{\hat{\imath}\hat{\jmath}}\mathfrak{u}_{kl})_{1 \leqslant k,l \leqslant n}$ and for $V = (V_{\hat{\jmath}\hat{\imath}})_{\hat{\jmath}\hat{\imath}}$ with $V_{\hat{\jmath}\hat{\imath}} := ({}_{\hat{\jmath}\hat{\imath}}\mathfrak{v}_{kl})_{1 \leqslant k,l \leqslant n}$. Since e_{1i}, δe_{1n}, and $e_{1i} + \delta e_{1n}$ are all rank-one nilpotents for $i = 2, \dots, n$, Lemma 2.5 gives $\Phi(e_{1i})\Phi(\delta e_{1n}) + \Phi(\delta e_{1n})\Phi(e_{1i}) = 0$. Recalling that

$$\Phi(e_{1i}) = \begin{pmatrix} \mathfrak{E}_{1i} & \\ & \mathfrak{E}_{i1} \end{pmatrix},$$

where $\mathfrak{E}_{ij}$ is a block-diagonal matrix, which has matrices e_{ij} on its main diagonal and zeros elsewhere, our last equation reads

$$\begin{pmatrix} \mathfrak{E}_{1i} & 0 \\ 0 & \mathfrak{E}_{i1} \end{pmatrix} \cdot \begin{pmatrix} A & U \\ V & B \end{pmatrix} + \begin{pmatrix} A & U \\ V & B \end{pmatrix} \cdot \begin{pmatrix} \mathfrak{E}_{1i} & 0 \\ 0 & \mathfrak{E}_{i1} \end{pmatrix} = 0 \quad (i = 2, \dots, n).$$

Writing down explicit equations on a particular block constituting A and U, one obtains

$$\begin{aligned} & e_{1i} \cdot A_{1\hat{\imath}} + A_{1\hat{\imath}} \cdot e_{1i} \\ & \quad = e_{1i} \cdot \begin{pmatrix} {}_{1\hat{\imath}}\mathfrak{a}_{11} & \cdots & {}_{1\hat{\imath}}\mathfrak{a}_{1n} \\ \vdots & \ddots & \vdots \\ {}_{1\hat{\imath}}\mathfrak{a}_{n1} & \cdots & {}_{1\hat{\imath}}\mathfrak{a}_{nn} \end{pmatrix} + \begin{pmatrix} {}_{1\hat{\imath}}\mathfrak{a}_{11} & \cdots & {}_{1\hat{\imath}}\mathfrak{a}_{1n} \\ \vdots & \ddots & \vdots \\ {}_{1\hat{\imath}}\mathfrak{a}_{n1} & \cdots & {}_{1\hat{\imath}}\mathfrak{a}_{nn} \end{pmatrix} \cdot e_{1i} = 0, \end{aligned}$$

on the one hand, and

$$\begin{aligned} & e_{1i} \cdot U_{1\hat{\jmath}} + U_{1\hat{\jmath}} \cdot e_{i1} \\ & \quad = e_{1i} \cdot \begin{pmatrix} {}_{1\hat{\jmath}}\mathfrak{u}_{11} & \cdots & {}_{1\hat{\jmath}}\mathfrak{u}_{1n} \\ \vdots & \ddots & \vdots \\ {}_{1\hat{\jmath}}\mathfrak{u}_{n1} & \cdots & {}_{1\hat{\jmath}}\mathfrak{u}_{nn} \end{pmatrix} + \begin{pmatrix} {}_{1\hat{\jmath}}\mathfrak{u}_{11} & \cdots & {}_{1\hat{\jmath}}\mathfrak{u}_{1n} \\ \vdots & \ddots & \vdots \\ {}_{1\hat{\jmath}}\mathfrak{u}_{n1} & \cdots & {}_{1\hat{\jmath}}\mathfrak{u}_{nn} \end{pmatrix} \cdot e_{i1} = 0 \end{aligned}$$

on the other; here, $i = 2, \dots, n$. An easy computation now reveals that the constituents of A and U must be of the form:

$$A_{1\hat{\imath}} = \left(\begin{array}{c:cccc} {}_{1\hat{\imath}}\mathfrak{a}_{11} & & {}_{1\hat{\imath}}\mathbf{a} & & \\ \hdashline & -({}_{1\hat{\imath}}\mathfrak{a}_{11}) & & & \\ \mathbf{0} & & & \ddots & \\ & & & & -({}_{1\hat{\imath}}\mathfrak{a}_{11}) \end{array}\right), \qquad {}_{1\hat{\imath}}\mathbf{a} = ({}_{1\hat{\imath}}\mathfrak{a}_{12}, \dots, {}_{1\hat{\imath}}\mathfrak{a}_{1n})$$

and

$$U_{1\hat{\jmath}} = \left(\begin{array}{c:c} {}_{1\hat{\jmath}}\mathfrak{u}_{11} & {}_{1\hat{\jmath}}\mathbf{u} \\ \hdashline -{}_{1\hat{\jmath}}\mathbf{u}^{\mathrm{t}} & \mathbf{0} \end{array}\right), \qquad {}_{1\hat{\jmath}}\mathbf{u} = ({}_{1\hat{\jmath}}\mathfrak{u}_{12}, \dots, {}_{1\hat{\jmath}}\mathfrak{u}_{1n}).$$

Now, as $e_{nn} + \delta e_{1n}$ is an idempotent, Eq. (6) (with e_{nn} and δe_{1n} in place of p and n, respectively) shows that $\Phi(e_{nn})\Phi(\delta e_{1n}) + \Phi(\delta e_{1n})\Phi(e_{nn}) = \Phi(\delta e_{1n})$. Repeating the previous arguments with this equation gives $A_{1\hat{\imath}} \equiv {}_{1\hat{\imath}}\mathfrak{a}_{1n} e_{1n}$, so that A has the desired form. Using same equation, combined with $\Phi(e_{2n})\Phi(\delta e_{1n}) + \Phi(\delta e_{1n})\Phi(e_{2n}) = 0$ yields $U_{1\hat{\jmath}} \equiv 0$ and thus $U = 0$. As for B and V, we may use a similar method with nilpotents e_{in}, δe_{1n}, and $e_{in}+\delta e_{1n}$ $(i=1,\ldots,n-1)$ in place of e_{1i}, δe_{1n}, and $e_{1i}+\delta e_{1n}$.

Finally, we shift our attention to an arbitrary rank-one nilpotent $m\in\mathscr{M}_n(\mathbb{F})$. There exists invertible $q\in\mathscr{M}_n(\mathbb{F})$ with $m=q^{-1}e_{1n}q$. Hence, we may repeat the story using rank-one matrices $\delta m+q^{-1}e_{1i}q=q^{-1}(\delta e_{1n}+e_{1i})q$ and $\delta m+q^{-1}e_{in}q=q^{-1}(\delta e_{1n}+e_{in})q$. $\square$

The previous lemma suggests that a natural way to describe Φ is with the help of tensors. The next one will justify this view.

Lemma 2.8. *Under the assumptions of Lemma* 2.7, *there exist additive mappings* $\varpi:\varDelta\to\mathrm{Hom}_{\varDelta'}(\hat{\mathscr{Y}})$, $\varsigma:\varDelta\to\mathrm{Hom}_{\varDelta'}(\hat{\mathscr{Z}})$ *such that for* $\delta\in\varDelta$, *and* $m:=e_{ij}$ *or* $m:=e_{ii}-e_{jj}(i\neq j)$, *one has*

$$\Phi(\delta m)=(\varpi(\delta)\otimes_{\mathbb{F}} m)\oplus(\varsigma(\delta)\otimes_{\mathbb{F}} m^{\mathrm{t}})\oplus 0.$$

Moreover, $\varpi(1)=\mathrm{Id}_{\hat{\mathscr{Y}}}$ *and* $\varsigma(1)=\mathrm{Id}_{\hat{\mathscr{Z}}}$.

Proof. The last claim was settled in diagonalization lemma. To prove the rest we notice that $e_{ii}-e_{jj}$ can be written as a sum of three rank-one nilpotents, namely $(e_{ii}-e_{jj})=(e_{ii}+e_{ij}-e_{ji}-e_{jj})+e_{ji}-e_{ij}$; hence Lemma 2.7 implies that $\mathscr{Y}$ and $\mathscr{Z}$ (cf. Remark 2.2) are invariant for $\Phi(\delta m)$. It suffices, therefore, to focus on the restriction $\Phi_{\mathscr{Y}}:=\Phi(\_)|_{\mathscr{Y}}$.

Lemma 2.7 implies that, for $i\neq j$ and $k\neq l$, the blocks of $\Phi_{\mathscr{Y}}(\delta e_{ij})$ and $\Phi_{\mathscr{Y}}(\delta e_{kl})$ are of the form $\xi_{\hat{\imath}\hat{\imath}}e_{ij}$ and $\zeta_{\hat{\imath}\hat{\imath}}e_{kl}$, respectively. We intend to show that actually, $\xi_{\hat{\imath}\hat{\imath}}\equiv\zeta_{\hat{\imath}\hat{\imath}}$: well, if $i=k$ or $j=l$ this follows immediately from Lemma 2.7 since then $e_{ij}+e_{kl}$ is a rank-one nilpotent. Similarly when $i\neq l$—we may in succession use the above trick on e_{ij} and e_{il} and finally on e_{il} and e_{kl}; likewise when $j\neq k$. Lastly, if $e_{kl}=e_{ij}^{\mathrm{t}}$ we form two rank-one nilpotents, $m_1:=e_{ii}+e_{ij}-e_{ji}-e_{jj}$ and $m_2:=e_{ii}-e_{ij}+e_{ji}-e_{jj}$. By Lemma 2.7, the blocks of $\Phi_{\mathscr{Y}}(\delta(e_{ii}-e_{jj}))=\Phi_{\mathscr{Y}}(\delta m_1)-\Phi_{\mathscr{Y}}(\delta e_{ij})+\Phi_{\mathscr{Y}}(\delta e_{ji})=\Phi_{\mathscr{Y}}(\delta m_2)+\Phi_{\mathscr{Y}}(\delta e_{ij})-\Phi_{\mathscr{Y}}(\delta e_{ji})$ are of the form

$$\begin{aligned}\left\langle\begin{matrix}\mu_{\hat{\imath}\hat{\imath}} & \nu_{\hat{\imath}\hat{\imath}}\\ \upsilon_{\hat{\imath}\hat{\imath}} & \varrho_{\hat{\imath}\hat{\imath}}\end{matrix}\right\rangle &= \left\langle\begin{matrix}\epsilon_{\hat{\imath}\hat{\imath}} & \epsilon_{\hat{\imath}\hat{\imath}}\\ -\epsilon_{\hat{\imath}\hat{\imath}} & -\epsilon_{\hat{\imath}\hat{\imath}}\end{matrix}\right\rangle - \left\langle\begin{matrix}0 & \xi_{\hat{\imath}\hat{\imath}}\\ 0 & 0\end{matrix}\right\rangle + \left\langle\begin{matrix}0 & 0\\ \zeta_{\hat{\imath}\hat{\imath}} & 0\end{matrix}\right\rangle\\ &= \left\langle\begin{matrix}\varepsilon_{\hat{\imath}\hat{\imath}} & -\varepsilon_{\hat{\imath}\hat{\imath}}\\ \varepsilon_{\hat{\imath}\hat{\imath}} & -\varepsilon_{\hat{\imath}\hat{\imath}}\end{matrix}\right\rangle + \left\langle\begin{matrix}0 & \xi_{\hat{\imath}\hat{\imath}}\\ 0 & 0\end{matrix}\right\rangle - \left\langle\begin{matrix}0 & 0\\ \zeta_{\hat{\imath}\hat{\imath}} & 0\end{matrix}\right\rangle,\end{aligned}$$

here,

$$\left\langle\begin{matrix}\alpha & \beta\\ \gamma & \delta\end{matrix}\right\rangle := \alpha e_{ii}+\beta e_{ij}+\gamma e_{ji}+\delta e_{jj}.$$

Comparing the corresponding entries in a field $\varDelta'$ with $\mathrm{char}(\varDelta')\neq 2$ yields $\xi_{\hat{\imath}\hat{\imath}}=\zeta_{\hat{\imath}\hat{\imath}}$, and $\nu_{\hat{\imath}\hat{\imath}}=0=\upsilon_{\hat{\imath}\hat{\imath}}$, and also $\mu_{\hat{\imath}\hat{\imath}}=\xi_{\hat{\imath}\hat{\imath}}=-\varrho_{\hat{\imath}\hat{\imath}}(=\epsilon_{\hat{\imath}\hat{\imath}}=\epsilon_{\hat{\imath}\hat{\imath}})$.

These results enable us to define the mapping $\varpi:\varDelta\to\mathrm{Hom}_{\varDelta'}(\hat{\mathscr{Y}})$ by $\varpi(\delta)\hat{\mathbf{y}}_{\hat{\imath}}\mapsto\sum_i\xi_{\hat{\imath}\hat{\imath}}\hat{\mathbf{y}}_i$, where $\hat{\mathbf{y}}_i$ are the same as in Remark 2.2 and $\xi_{\hat{\imath}\hat{\imath}}$ are the only nonzero entries in the $(\imath\hat{\imath})$th block of $\Phi_{\mathscr{Y}}(\delta m)$ with, say, $m = e_{12}$. Of course, the sum is finite since only finitely many $\xi_{\imath\hat{\imath}}$ are nonzero with $\hat{\imath}$ fixed (see last claim of Lemma 2.3). It easily follows that $\Phi_{\mathscr{Y}}(\delta m) = \varpi(\delta) \otimes_{\mathbb{F}} m$, as anticipated. $\square$

This renders possible the following characterization of additive idempotence-preservers.

Theorem 2.9. *Let $n \geqslant 3$. There exists a unique Jordan homomorphism $\breve{\Phi} : \mathscr{M}_n(\Delta) \to \mathscr{B}$ such that $\breve{\Phi}|_{\mathscr{P}} \equiv \Phi|_{\mathscr{P}}$.*

Proof. To show the existence; Lemma 2.8 suggests that the natural definition is

$$\breve{\Phi} : \delta e_{ij} \mapsto (\varpi(\delta) \otimes_{\mathbb{F}} e_{ij}) \oplus (\varsigma(\delta) \otimes_{\mathbb{F}} e_{ij}^{\mathrm{t}}) \oplus 0. \tag{7}$$

It remains to show that this is Jordan, that it equals Φ on $\mathscr{P}$, and that it is unique. This will be done in three separate steps.

Step 1. To see that $\breve{\Phi}$ is Jordan it is enough to check that ϖ and ς are homomorphisms. Assume $\varpi \neq 0$, pick arbitrary, $\alpha, \beta \in \Delta$, and note that the rank-one nilpotent $m_{\alpha,\beta} := \alpha(e_{11} - e_{22}) + \beta e_{12} - (\alpha^2/\beta)e_{21}$ is mapped into a nilpotent of a nilindex 2 or less. Furthermore, from the previous Lemma we have $\Phi(m_{\alpha,\beta}) = \breve{\Phi}(m_{\alpha,\beta})$. Hence,

$$\begin{aligned}\Phi(m_{\alpha,\beta})|_{\mathscr{Y}} &= \breve{\Phi}(m_{\alpha,\beta})|_{\mathscr{Y}}\\ &= \varpi(\alpha) \otimes (e_{11} - e_{22}) + \varpi(\beta) \otimes e_{12} - \varpi\left(\frac{\alpha^2}{\beta}\right) \otimes e_{21}\end{aligned}$$

which imply that

$$\begin{aligned}0 = (\Phi(m_{\alpha,\beta})|_{\mathscr{Y}})^2 &= \left(\varpi(\alpha)^2 - \varpi(\beta)\varpi\left(\frac{\alpha^2}{\beta}\right)\right) \otimes e_{11}\\ &\quad + \left(\varpi(\alpha)^2 - \varpi\left(\frac{\alpha^2}{\beta}\right)\varpi(\beta)\right) \otimes e_{22}\\ &\quad + (\varpi(\alpha)\varpi(\beta) - \varpi(\beta)\varpi(\alpha)) \otimes e_{12}\\ &\quad + \left(\varpi(\alpha)\varpi\left(\frac{\alpha^2}{\beta}\right) - \varpi\left(\frac{\alpha^2}{\beta}\right)\varpi(\alpha)\right) \otimes e_{21}.\end{aligned}$$

Inserting $\beta := 1$ in the first summand shows ϖ is Jordan. Thus, $\varpi((\alpha+\beta)^2) = (\varpi(\alpha) + \varpi(\beta))^2$ which, together with the fact that the third term in the above equation equals zero, gives that ϖ is a homomorphism. Similarly for ς.

Step 2. Let us show that $\breve{\Phi}$ equals Φ on $\mathscr{P}$. As any idempotent is a sum of (pairwise orthogonal) rank-one idempotents—think of the matrix representation in a suitable basis—it suffices to show that $\breve{\Phi}(p) = \Phi(p)$ for any $p = p^2$ of the rank one. Such p equals

$$p = \sum_{i,j} \delta_{ij} e_{ij} = \sum_i \delta_{ii}(e_{ii} - e_{11}) + \sum_i \delta_{ii} e_{11} + \sum_{i \neq j} \delta_{ij} e_{ij}$$

for some $\delta_{ij} \in \varDelta$ and with the trace, $\mathrm{Tr}\, p := \sum_i \delta_{ii} = 1$. Now, as $\breve{\varPhi}$ and $\varPhi$ agree on $\delta_{ij} e_{ij}$ for $i \neq j$, on $\delta_{ii}(e_{ii} - e_{11})$, and on $e_{11} = \sum_i \delta_{ii} e_{11}$ (see diagonalization lemma), one has $\varPhi(p) = \breve{\varPhi}(p)$, as claimed.

Step 3. Finally, we settle the uniqueness: as δe_{ij} is already in $\mathscr{P}$ for $i \neq j$ and $\delta \in \varDelta$ it is enough to see that the Jordan mapping $\breve{\varPhi}$ can be uniquely defined on the elements of the form δe_{ii}. This, however, follows easily: namely, $\delta e_{ii} = 1/2[(\delta e_{ij} + e_{ji})^2 + \delta(e_{ii} + e_{ij} - e_{ji} - e_{jj}) - \delta e_{ij} + \delta e_{ji}]$, and the summands on the right side are in $\mathscr{P}$ (where $\varPhi$ and $\breve{\varPhi}$ must agree), while $\breve{\varPhi}$ is Jordan. $\square$

Remark 2.10. The theorem fails when $n = 1$; a simple counterexample is furnished with an additive, unital $\varPhi : \mathbb{R} \to \mathbb{Q}$ (that maps reals to rationals). If it is Jordan, it would necessarily be a homomorphism. However, there are no nontrivial ones. Surprisingly though, it fails also when $n = 2$, as shown in Example 2.12.

The last assertion of this section is an extension of the theorem above. We recall that the algebra $\mathscr{A}$ is called *a local matrix algebra* if the arbitrary finite subset can be embedded in a subalgebra, isomorphic to some $\mathscr{M}_n(\varDelta)$ (see also [7]). Any such subalgebra is of course finitely generated. Therefore, the immediate consequence of the definition is that, given two such subalgebras, there exists a third one, containing them both.

Corollary 2.11. *Suppose $\varDelta, \varDelta'$ are fields with $\mathrm{char}(\varDelta) \neq 2$, $\mathscr{A} \notin \{\varDelta, \mathscr{M}_2(\varDelta)\}$ a local matrix algebra over $\varDelta$, and $\mathscr{C}$ an algebra over $\varDelta'$. If $\varPhi : \mathscr{A} \to \mathscr{C}$ is an additive, idempotence-preserving mapping, then there exists a unique Jordan mapping $\breve{\varPhi} : \mathscr{A} \to \mathscr{C}$ that agrees with $\varPhi$ on $\mathscr{P}(\mathscr{A})$. Moreover, $\breve{\varPhi}$ is a direct sum of the homomorphism and the antihomomorphism.*

Proof. Assume without loss of generality that $\mathscr{C} = \mathscr{B} := \mathrm{Hom}_{\Delta'}(\mathscr{X})$. For arbitrary two elements $a, b \in \mathscr{A}$ there exists a subalgebra $\mathscr{A}_1 \leqslant \mathscr{A}$, isomorphic to $\mathscr{M}_n(\varDelta)$ for some integer $n \geqslant 3$, and with $a, b \in \mathscr{A}_1$. We then use the above theorem to find the Jordan extension $\breve{\varPhi} : \mathscr{A}_1 \to \mathscr{C}$ of $\varPhi|_{\mathscr{A}_1}$. Considering the part on the uniqueness of the theorem above, these mappings compose the desired one.

The last assertion follows by [7, Theorem 8]. $\square$

We finish with an example, showing the failure of Theorem 2.9 when $n = 2$.

Example 2.12. Suppose $\varDelta$ is a Galois field $\mathrm{Gf}(3^2)$. It is trivial to see that $\varDelta$ is generated by the element a, subject to conditions $\mathrm{a}^2 = \mathrm{a} + 1$ and $\mathrm{a} + \mathrm{a} + \mathrm{a} = 0$. We define an additive mapping $\varPhi : \mathscr{M}_2(\varDelta) \to \mathscr{M}_4(\varDelta)$ by

$$\begin{aligned}\Phi : \delta e_{11} &\mapsto \delta(e_{11} + e_{33})\\ \delta e_{22} &\mapsto \delta(e_{22} + e_{44})\\ \delta e_{21} &\mapsto \delta(e_{21} + e_{34})\\ (i + j\mathsf{a})e_{12} &\mapsto (i + j\mathsf{a}) \cdot (e_{12} + e_{43}) + j \cdot (e_{32} - e_{41}) \quad (i, j \in \mathbb{Z}_3, \delta \in \Delta).\end{aligned}$$

Straightforward, yet tedious calculations show that Φ preserves all 92 idempotents in $\mathscr{M}_2(\Delta)$. However, $\Phi|_{\mathscr{P}}$ has no Jordan extension: suppose, to reach a contradiction, that $\breve{\Phi}$ is the one. As $\mathscr{M}_2(\Delta)$ is a matrix ring, $\breve{\Phi}$ must be a direct sum of the homomorphism ϕ and the antihomomorphism τ (see [7, Theorem 7]). Let $P := \phi(\mathrm{Id})$; obviously, P commutes with every $\Phi(m)$, $m \in \mathscr{M}_2(\mathbb{F})$; in particular, $P\Phi(e_{11}) = \Phi(e_{11})P$, and $P\Phi(e_{12}) = \Phi(e_{12})P$, and $P\Phi(e_{21}) = \Phi(e_{21})P$. So, by putting $P = \sum_{i,j=1}^{4} \delta_{ij} e_{ij}$ into these equations, it follows easily that $P = \delta_{11}(e_{11} + e_{22}) + \delta_{44}(e_{33} + e_{44})$. Furthermore, $P^2 = P$ implies $\delta_{11}, \delta_{44} \in \{0, 1\}$. Well, $\delta_{11} \neq \delta_{44}$, as this would imply that $\breve{\Phi}$ is an (anti)homomorphism, contradicting $\breve{\Phi}(e_{12})\breve{\Phi}(e_{11}) = \Phi(e_{12})\Phi(e_{11}) = e_{43} \neq \breve{\Phi}(e_{12}e_{11})$ or $\breve{\Phi}(e_{11}e_{12})$. Thus, either $\delta_{11} = 0$ and $\delta_{44} = 1$, or the other way around. Consequently, $\breve{\Phi}$ would be block diagonal, contradicting the definition of $\Phi(\delta e_{12}) = \breve{\Phi}(\delta e_{12})$.

3. Mappings defined on $\mathscr{B}(\mathscr{X})$

We shift our attention to additive idempotence-preservers between the algebras $\mathscr{B}(\mathscr{X})$ and $\mathscr{B}(\mathscr{Y})$ of bounded operators on real or complex *infinite dimensional Banach space*. The results of the previous section will be used on the subalgebra $\mathscr{F}(\mathscr{X})$ of all *finite-rank* operators. It is known that this is a local matrix algebra (see [7, Theorem 9]), thus Corollary 2.11 gives us a unique Jordan extension $\breve{\Phi}$ of $\Phi|_{\mathscr{P}_{\mathrm{F}}}$, where $\mathscr{P}_{\mathrm{F}} := \mathrm{Lin}_{\mathbb{Q}}\{p \in \mathscr{F}(\mathscr{X});\, p^2 = p\}$ and $\mathbb{Q}$ is a field of rationals. We nevertheless emphasize that $\breve{\Phi}$ and Φ may well disagree on the set $\mathscr{F}(\mathscr{X}) \setminus \mathscr{P}_{\mathrm{F}}$.

From now on, $\mathscr{P} := \mathrm{Lin}_{\mathbb{Q}}\{p \in \mathscr{B}(\mathscr{X});\, p^2 = p\}$, the adjoint of operator $T : \mathscr{X} \to \mathscr{Y}$ is denoted by $T^* : \mathscr{Y}^* \to \mathscr{X}^*$, and—similarly to the previous section—for $\mathbf{x} \in \mathscr{X}$ and $f \in \mathscr{X}^*$, $\mathbf{x} \otimes f$ is a *bounded*, rank-one operator that maps $\mathbf{z} \in \mathscr{X}$ to $f(\mathbf{z})\mathbf{x} \in \mathscr{X}$.

Lemma 3.1. *Suppose $\mathscr{C}$ is an algebra over $\mathbb{K}(= \mathbb{R}$ or $\mathbb{C})$ and $\Phi : \mathscr{B}(\mathscr{X}) \to \mathscr{C}$ an additive mapping, which preserves idempotents. Then, for arbitrary idempotent P and arbitrary rank-one element a in $\mathscr{B}(\mathscr{X})$, the following holds true*:
1. *If $Pa = 0 = aP$, then $\breve{\Phi}(a)\Phi(P) = 0 = \Phi(P)\breve{\Phi}(a)$.*
2. *If $Pa = a = aP$, then $\breve{\Phi}(a)\Phi(P) = \breve{\Phi}(a) = \Phi(P)\breve{\Phi}(a)$.*

Proof. In the first claim, if $a = \lambda p$ with $\lambda \in \mathbb{K}$ and p a rank-one idempotent, then $a = pap$, so $\breve{\Phi}(a) = \breve{\Phi}(p)\breve{\Phi}(a)\breve{\Phi}(p)$ by Eq. (1). Since $P + p$ is also idempotent, an easy calculation reveals $\Phi(p)\Phi(P) = 0 = \Phi(P)\Phi(p)$ (see also [1, p. 922]), giving $\breve{\Phi}(a)\Phi(P) = 0 = \Phi(P)\breve{\Phi}(a)$.

Otherwise, $a = \mathbf{x} \otimes f$ is a nonzero nilpotent with $P\mathbf{x} = 0 = P^* f$. Therefore, $0 \neq f = (\mathrm{Id} - P)^* f$, so there exists a vector $\mathbf{z} \in \mathrm{Im}(\mathrm{Id} - P) = \mathrm{Ker} P$ with $f(\mathbf{z}) = 1$. It gives us two idempotents $p_1 := (\mathbf{x} + \mathbf{z}) \otimes f$ and $p_2 := \mathbf{z} \otimes f$ such that $a = p_1 - p_2$ and $P p_i = 0 = p_i P (i = 1, 2)$. The above arguments imply that $\breve{\Phi}(p_i)\Phi(P) = 0 = \Phi(P)\breve{\Phi}(p_i)$ hence also $\breve{\Phi}(a)\Phi(P) = 0 = \Phi(P)\breve{\Phi}(a)$.

In the last claim, if $a = p$, then $(P - p)$ and p are orthogonal idempotents, and so are their Φ-images. Therefore, $\Phi(P)\Phi(p) = \Phi(p) = \Phi(p)\Phi(P)$. The rest follows as before. $\square$

We can finally prove the generalization of [2, Theorem 1] (see also Remark 3.4).

Main Theorem. *Let $\Phi : \mathscr{B}(\mathscr{X}) \to \mathscr{B}(\mathscr{Y})$ be an additive mapping that preserves idempotents. Suppose further that $\Phi(\mathscr{P}_{\mathrm{F}}) \neq 0$ and that $\Phi(\mathscr{P})$ contains* all minimal idempotents. *Then, there exists a unique Jordan extension $\tilde{\Phi} : \mathscr{B}(\mathscr{X}) \to \mathscr{B}(\mathscr{Y})$ of $\Phi|_{\mathscr{P}}$. It takes one of the two following forms*:

$$A \mapsto TAT^{-1}, \tag{i}$$

$$A \mapsto TA^*T^{-1} \tag{ii}$$

for some continuous, (conjugate) linear bijection $T : \mathscr{X} \to \mathscr{Y}$, or $T : \mathscr{X}^* \to \mathscr{Y}$, respectively. In the latter case $\mathscr{X}$ as well as $\mathscr{Y}$ are reflexive.

Proof. First we settle the existence. Let $\breve{\Phi} = \phi \dot{\oplus} \tau : \mathscr{F}(\mathscr{X}) \to \mathscr{B}(\mathscr{Y})$ be a unique Jordan extension of $\Phi|_{\mathscr{P}_{\mathrm{F}}}$, as guaranteed by Corollary 2.11. Since $\mathscr{F}(\mathscr{X})$ is a simple ring, it forces each of ϕ and τ to either vanish identically, or else to be injective. However, as $\Phi(\mathscr{P}_{\mathrm{F}}) \neq 0$, at least one of the mappings ϕ and τ is nonzero; our main aim in the sequel is to show that *precisely one* is. This and the rest of the claims will be proved in consecutive steps.

Step 1. We note that for each idempotent $P \in \mathscr{B}(\mathscr{X})$ the subring $K_P := \mathbb{Q}P + \mathscr{F}(\mathscr{X})$ of $\mathscr{B}(\mathscr{X})$ admits a unique additive Jordan extension $\breve{\Phi}_P$ of $\breve{\Phi}$, if we define $\breve{\Phi}_P(P) := \Phi(P)$. To see this it is sufficient to show that

$$\breve{\Phi}(Pn + nP) = \breve{\Phi}_P(P)\breve{\Phi}(n) + \breve{\Phi}(n)\breve{\Phi}_P(P) \tag{8}$$

holds for arbitrary $n = \mathbf{x} \otimes f$ of the rank one. In fact, we may split $\mathbf{x} = P\mathbf{x} + (\mathrm{Id} - P)\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ and similarly $f = P^* f + (\mathrm{Id} - P)^* f = f_1 + f_2$, and then invoke Lemma 3.1 to see that $\breve{\Phi}_P(P)\breve{\Phi}(\mathbf{x}_1 \otimes f_1) = \breve{\Phi}(\mathbf{x}_1 \otimes f_1) = \breve{\Phi}(\mathbf{x}_1 \otimes f_1)\breve{\Phi}_P(P)$, and $\breve{\Phi}_P(P)\breve{\Phi}(\mathbf{x}_2 \otimes f_2) = 0 = \breve{\Phi}(\mathbf{x}_2 \otimes f_2)\breve{\Phi}_P(P)$. This manifestly implies the validity of (8) with $\mathbf{x}_i \otimes f_i$; $(i = 1, 2)$ in place of n. The only thing left to check is Eq. (8) against the cases $n_{12} = \mathbf{x}_1 \otimes f_2$ and $n_{21} = \mathbf{x}_2 \otimes f_1$. We consider the first one only. Namely, as $n_{12} = P\mathbf{x} \otimes (\mathrm{Id} - P)^* f$, it forces $n_{12}^2 = 0$ and $P + n_{12}$ to be an idempotent. Since Φ and $\breve{\Phi}_P$ agree on P, as well as on n_{12}, and as Φ preserves idempotents, we have

$$\begin{aligned}\check{\Phi}_P(P)+\check{\Phi}(n_{12}) &= (\check{\Phi}_P(P)+\check{\Phi}(n_{12}))^2\\ &= \check{\Phi}_P(P)+\check{\Phi}_P(P)\check{\Phi}(n_{12})+\check{\Phi}(n_{12})\check{\Phi}_P(P)+\Phi(n_{12})^2.\end{aligned}$$

Lemma 2.5 implies $\Phi(n_{12})^2=0$, therefore $\check{\Phi}(Pn_{12}+n_{12}P)=\check{\Phi}(n_{12}+0)=\check{\Phi}(n_{12})\check{\Phi}_P(P)+\check{\Phi}_P(P)\check{\Phi}(n_{12})$, which concludes the proof of Eq. (8).

Step 2. Next we claim that if we choose another idempotent Q, one has

$$\check{\Phi}(PaQ+QaP)=\Phi(P)\check{\Phi}(a)\Phi(Q)+\Phi(Q)\check{\Phi}(a)\Phi(P)\quad (a\in\mathscr{F}(\mathscr{X})).\tag{9}$$

It is of course enough to prove (9) for rank-one elements $a=\mathbf{x}\otimes f$. As before, we are facing four different possibilities; in the first one we assume that $\mathbf{x}\in\operatorname{Ker}P$ and $f\in\operatorname{Ker}P^*$. In this situation, the left-hand side of (9) equals $\check{\Phi}(0+0)=0$. However, by Lemma 3.1, $\Phi(P)\check{\Phi}(a)=0=\check{\Phi}(a)\Phi(P)$, so (9) holds. Similarly in the second case when $\mathbf{x}\in\operatorname{Im}P$ and $f\in\operatorname{Im}P^*$: then, the left-hand side reads $\check{\Phi}(aQ+Qa)$, and this in turn equals $\check{\Phi}(a)\Phi(Q)+\Phi(Q)\check{\Phi}(a)$, since $\check{\Phi}_Q$ is Jordan. Once again, by using Lemma 3.1 we establish the validity of Eq. (9).

In the third case we assume $\mathbf{x}\in\operatorname{Im}P$ and $f\in\operatorname{Ker}P^*$. Now, if rank $P<\infty$ then (9) holds since $\check{\Phi}_Q$ is Jordan (see Eq. (2)). Thus, we may well assume that rank $P=\infty$. Pick arbitrary functional g with $g(\mathbf{x})=1$, and set $g_0:=P^*g$. Obviously, $g_0(\mathbf{x})=g(\mathbf{x})=1$ implying that $p_0:=\mathbf{x}\otimes g_0$ is a minimal idempotent with $Pp_0=p_0=p_0P$. Hence, $\tilde{P}:=P-p_0$ is also an idempotent, and a simple computation yields $\tilde{P}\mathbf{x}=0=\tilde{P}^*f$. But then it was already shown above that $\check{\Phi}(\tilde{P}aQ+Qa\tilde{P})=\Phi(\tilde{P})\check{\Phi}(a)\Phi(Q)+\Phi(Q)\check{\Phi}(a)\Phi(\tilde{P})$. Since Eq. (9) is also valid with p_0 in place of P (namely, $\check{\Phi}_Q$ is Jordan), it clearly follows that Eq. (9) is sound in this case. We proceed similarly in the remaining possibility when $\mathbf{x}\in\operatorname{Ker}P$ and $f\in\operatorname{Im}P^*$, which completely proves Eq. (9).

Step 3. By the assumptions, there exists some rational λ_i such that $A=\sum_1^n\lambda_iP_i\in\mathscr{P}$ is mapped into a minimal idempotent. By (9), and as $\check{\Phi}_{P_i}$ is Jordan, one has

$$\check{\Phi}(AnA)=\Phi(A)\check{\Phi}(n)\Phi(A)\quad (n\in\mathscr{F}(\mathscr{X})).\tag{10}$$

It is straightforward to find a rank-one operator n with $AnA\neq 0$; then, injectivity of $\check{\Phi}$ forces the left (thus also the right)-hand side of (10) to be nonzero. Further, as $\Phi(A)$ is a minimal idempotent the right-hand side equals $\xi\Phi(A)$—it is therefore of the rank one. The same must be true of $\check{\Phi}(AnA)=\phi(AnA)\dot{\oplus}\tau(AnA)$; consequently, either $\psi(AnA)=0$ or $\tau(AnA)=0$. Therefore, either $\psi\equiv 0$ or else $\tau\equiv 0$, as claimed.

Step 4. We suppose that $\check{\Phi}=\tau$ is an antihomomorphism; the instance when it is a homomorphism can be dealt with similarly and is even more straightforward.

By (10) the antihomomorphism τ maps rank-one operator *AnA* into a rank-one $\xi\Phi(A)$. Therefore, τ preserves minimal idempotents since they can be written as $p=q_1\cdot(AnA)\cdot q_2$ for $q_1,q_2\in\mathscr{F}(\mathscr{X})$. Inversely, as $\Phi(\mathscr{P})$ contains *all minimal idempotents on* $\mathscr{Y}$, they are all in $\operatorname{Im}\tau$ by Eq. (10). Thus, $\operatorname{Im}\tau=\mathscr{F}(\mathscr{Y})$ since any rank-one operator on $\mathscr{Y}$ is a product of two minimal idempotents. Lastly, if p is a minimal idempotent and λ a scalar then $\lambda p = p \cdot (\lambda p) \cdot p$, which implies that $\tau(\lambda p) = \tau(p)\tau(\lambda p)\tau(p) \in \mathbb{K}\tau(p)$ (i.e., τ preserves 'linear spans of minimal idempotents'). Now, it was shown in [9, Main Theorem] that there exists a continuous, (conjugate) linear bijection $T : \mathscr{X}^* \to \mathscr{Y}$ with

$$\check{\Phi}(n) = \tau(n) = Tn^*T^{-1} \quad (n \in \mathscr{F}(\mathscr{X})). \tag{11}$$

We claim that (11) is actually valid for an arbitrary idempotent P in place of n, and thus, by additivity, for an arbitrary element of $\mathscr{P}$. To see this we define an auxiliary function $S := TP^* - \Phi(P)T : \mathscr{X}^* \to \mathscr{Y}$, and suppose $Sf \neq 0$ for some $f \in \mathscr{X}^*$. Furthermore, suppose with no loss of generality that either $P^*f = f$ or else $P^*f = 0$. In the first case we can find a minimal idempotent q with $Pq = q = qP$ and $q^*f = f$. Then, however, $Sq^*f = Sf \neq 0$, while by (11) and Lemma 3.1.

$$Sq^* = TP^*q^* - \Phi(P)Tq^* = T(qP)^* - \Phi(P)\check{\Phi}(q)T = Tq^* - \check{\Phi}(q)T = 0,$$

a contradiction. Similarly, if $P^*f = 0$, we can find a minimal idempotent q orthogonal to P and with $q^*f = f$, to come to a contradiction by analogous arguments. It follows that $S \equiv 0$ and hence $\Phi(P) = TP^*T^{-1}$ for every $P \in \mathscr{P}$. (We remark that this idea was borrowed from [2].)

Step 5. As for the reflexivity of $\mathscr{X}$ and $\mathscr{Y}$, we may view them as real Banach spaces, and then apply the idea from [3] as follows: a rank-one operator $\mathbf{x} \otimes f \in \mathscr{B}(\mathscr{X})$ is mapped into $\tau(\mathbf{x} \otimes f) = Tf \otimes (T^{-1})'\kappa\mathbf{x}$, where $\kappa : \mathscr{X} \hookrightarrow \mathscr{X}''$ is a natural embedding and $'$ denotes an adjunction in the *real Banach spaces*. Since all rank-one operators on $\mathscr{Y}$ are of this form, the mapping $(T^{-1})'\kappa : \mathscr{X} \to \mathscr{Y}'$ is bijective. This in turn implies that κ is bijective, hence $\mathscr{X}$ is reflexive as a real Banach space, and consequently as a complex one. The same is true of $\mathscr{Y}$ since T^{-1} maps $\mathscr{Y}$ (conjugate) isomorphically onto $\mathscr{X}'$.

Step 6. Let us finally address the uniqueness of Jordan extensions. If $\tilde{\tilde{\Phi}}$ is another one, we may form an additive Jordan mapping $\Psi : \mathscr{B}(\mathscr{X}) \to \mathscr{B}(\mathscr{X})$ by

$$\Psi : A \mapsto \kappa^{-1}(T^{-1}\tilde{\tilde{\Phi}}(A)T)^*\kappa. \tag{12}$$

By assumptions, $\Psi(P) = \kappa^{-1}(T^{-1}\tilde{\tilde{\Phi}}(P)T)^*\kappa = \kappa^{-1}(T^{-1}\Phi(P)T)^*\kappa = P$ for every $P \in \mathscr{P}$. So suppose $\Psi(A) \neq A$ for some $A \in \mathscr{B}(\mathscr{X})$. As Ψ is Jordan and as there is but one Jordan extension of $\Psi|_{\mathscr{P}_\mathrm{F}}$ onto $\mathscr{F}(\mathscr{X})$, we have $\Psi|_{\mathscr{F}(\mathscr{X})} = \mathrm{Id}_{\mathscr{F}(\mathscr{X})}$. Hence,

$$\begin{aligned} An + nA &= \Psi(An + nA) = \Psi(A)\Psi(n) + \Psi(n)\Psi(A) \\ &= \Psi(A)n + n\Psi(A) \quad (n = \mathbf{x} \otimes f \in \mathscr{F}(\mathscr{X})), \end{aligned}$$

which implies that $(\Psi(A) - A)\mathbf{x} \otimes f = \mathbf{x} \otimes (A - \Psi(A))^*f$ holds for all $\mathbf{x} \in \mathscr{X}$, $f \in \mathscr{X}^*$. As $\Psi(A) \neq A$, this immediately yields that $(\Psi(A) - A)\mathbf{x}$ and $\mathbf{x}$ are linearly dependent for all $\mathbf{x}$, hence $\Psi(A) - A = \lambda_A \mathrm{Id}$ for some scalar λ_A. Then, however, $A^2 + \lambda_{A^2} \cdot \mathrm{Id} = \Psi(A^2) = \Psi(A)^2 = (A + \lambda_A \cdot \mathrm{Id})^2$, so that $\lambda_A \neq 0$ only when A is a scalar, i.e., $A = \mu\mathrm{Id}$. Even in this instance we can take arbitrary $n := \mathbf{x} \otimes f$, and calculate

$$\begin{aligned} 2\mu n &= \Psi(2\mu n) = \Psi((\mu\mathrm{Id})n + n(\mu\mathrm{Id})) = \Psi(\mu\mathrm{Id})\Psi(n) + \Psi(n)\Psi(\mu\mathrm{Id}) \\ &= (\mu\mathrm{Id} + \lambda_{(\mu\mathrm{Id})}\mathrm{Id})n + n(\mu\mathrm{Id} + \lambda_{(\mu\mathrm{Id})}\mathrm{Id}) = 2(\mu + \lambda_{(\mu\mathrm{Id})})n \end{aligned}$$

thus $\lambda_{(\mu\mathrm{Id})} = 0$ and consequently $\Psi(A) - A = \lambda_A \equiv 0$, wherefrom the result follows. $\square$

Remark 3.2. The inclusion mapping $\mathscr{B}(\ell^2) \hookrightarrow \mathscr{B}(\ell^2 \otimes \ell^2)$ shows that parts of the Theorem are no longer valid if the assumption that $\Phi(\mathscr{P})$ contains *all minimal idempotents* is relaxed.

If $\mathscr{X}$ is a Hilbert space, we can say a bit more.

Corollary 3.3. *Suppose $\mathscr{X}$ is a Hilbert space and $\Phi : \mathscr{B}(\mathscr{X}) \to \mathscr{B}(\mathscr{Y})$ an additive, surjective, idempotence-preserving map. Then, Φ either annihilates minimal idempotents, or else there exists a (conjugate) linear, invertible $T \in \mathscr{B}(\mathscr{X})$ such that $\Phi(a) = TaT^{-1}$ or $\Phi(a) = Ta^*T^{-1}$, respectively. Here, a^* is a Hilbert-space adjoint of a.*

Proof. By [10], each operator on $\mathscr{X}$ is a sum of five idempotents, implying that $\mathscr{P} = \mathscr{B}(\mathscr{X})$. Hence, the assumptions of the Main Theorem are fulfilled and the result follows easily. $\square$

Remark 3.4. In general Banach spaces the situation is a bit different. Namely, there exists a Banach space $\mathscr{X}$ such that $\mathscr{B}(\mathscr{X})$ has a nontrivial multiplicative functional γ (see [8, Example 1.d.2] and [13]). So, $\mathscr{P} \underset{\neq}{\subset} \mathscr{B}(\mathscr{X})$ (otherwise $\mathrm{Im}\gamma \subset \mathbb{Q}$, contradicting linearity). We may, however, split $\mathscr{B}(\mathscr{X}) = \mathscr{P} \oplus \mathscr{Q}$ into $\mathbb{Q}$ submodules; hence any Φ from the theorem is of the form $\Phi = \tilde{\Phi} + \Xi\mathfrak{P}$ with $\tilde{\Phi}$ a (conjugate) linear Jordan isomorphism and $\Xi : \mathscr{Q} \to \mathscr{B}(\mathscr{Y})$ an additive mapping, and $\mathfrak{P} : \mathscr{P} \oplus \mathscr{Q} \to \mathscr{Q}$ an additive projection.

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